Abstract—This paper deals with different beamforming techniques for DOA estimation. High resolution techniques such as Multiple Signal Classification (MUSIC) and Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) which are subspace based techniques are also discussed. Finally, further resolution improvement is achieved through the use of directional sensors. The computational complexities of beamforming techniques are also compared.

Index Terms—Beamforming, computations, direction-of-arrival, multiple, radar, resolution.

I. INTRODUCTION

Beamforming uses an array of receivers (sensors) for directional transmission/reception. The main goal for formation of an array and array processing is to combine the sensors outputs so that the SNR can be enhanced, information about the number of sources/targets and direction of each can be determined, various parameters of the incident signals can be estimated.

The main requirement in many source localization applications e.g. Radar, Sonar is to estimate the direction of arrival without errors. When two sources have a small angular distance between them in space, angular resolution is an important area to concern about; otherwise some sources/targets would not be detected. The objective of paper is to discuss DOA estimation algorithms with focus on the enhancement of angular resolution. The requirement is achieve a higher resolution with minimum number of computations.

Propagating signals contain much information about the sources that produce them. Not only does each signal’s waveform express the nature of the source, its temporal and spatial characteristics combined with the laws of physics allow us to determine the source’s location [1]. For propagating signals, more is needed; spatiotemporal filtering must be employed to separate signals according to their directions of propagation and their frequency content.

II. MODEL FOR INCIDENT WAVES

A uniform linear array (ULA) is used for beamforming. The sources/targets are assumed to be in the far-field region so that the waves coming from them to the ULA can be considered to be plane waves. The data from the ULA is

\[ x(n) = A(\phi)s(n) + n(n) \]  

Here, \( A(\phi) \) (\( N \times M \)) is the steering matrix where \( N \) is the number of sensors and \( M \) is the number of sources. \( s(n) \) is the signals vector and \( n(n) \) is additive white Gaussian noise [2]. The number of snapshots of the signals are \( K, n=1, 2, 3, \ldots, K \).

III. BEAMFORMING TECHNIQUES

Beamforming techniques can be divided into spectral estimation techniques and subspace based methods [3]. In spectral estimation, a spectrum-like function of the parameter of interest e.g. the DOA is formed. The locations of the highest (separated) peaks of the function are the DOA estimates. The idea is to steer the array in one direction at a time and measure the output power. The steering locations which result in maximum power yield the DOA estimates.

The beamformer’s output power is [3]

\[ P(w) = w^H \hat{R}w \]  

A. Conventional Delay and Sum Beamformer (CBF)

It maximizes the power of the beamformer output for a given input signal. For CBF, weight vector is the steering
vector. For different angles, the output power is measured [3]

\[ P(\phi) = a^H(\phi)\hat{R}a(\phi) \]  

(4)

where the steering vector

\[ a(\phi) = g(\phi)\left[ e^{-j\beta \sin \phi} e^{-j2\beta \sin \phi} \ldots e^{-j(N-1)\beta \sin \phi}\right]^T \]  

(5)

And \( g(\phi) = 1 \), for omnidirectional sensors. Following plot shows the CBF output power when 2 sources are present at \( \phi_1 = -30^\circ \) and \( \phi_2 = 20^\circ \)

From the above figure, it can be seen that the resolution of MVDR is much better than that of CBF.

Now the high resolution subspace based methods are discussed which are based on the decomposition of correlation matrix into signal and noise subspaces.

\[ R = A\Pi A^H + \sigma^2 I = U_s \Lambda_s U_s^H + U_n \Lambda_n U_n^H \]  

(7)

C. Multiple Signal Classification (MUSIC)

This algorithm uses the fact that all noise eigenvectors are orthogonal to the signal steering vectors [5]

\[ U_n^H a(\phi) = 0, \quad \phi \in \{\phi_1, \phi_2, \ldots, \phi_d\} \]  

(8)

The output spectrum for MUSIC beamformer is

\[ P(\phi) = \frac{1}{|U_n^H a(\phi)|^2} = \frac{1}{\hat{R}a(\phi)U_s^H U_s a(\phi)} \]  

(9)

In the output spectrum, \( M \) largest peaks correspond to DOAs

A limitation of the CBF is that it cannot resolve 2 targets within the beamwidth. Consider 2 sources at \( 0^\circ \) and \( 20^\circ \)

One way to increase the resolution is to increase the number of sensors, which reduces the beamwidth. But it also increases the cost of the beamformer.

B. Minimum Variance Distortionless Response Beamformer (MVDR)

To reduce the limitations of conventional beamformer, such as to increase the resolving power of two sources spaced closer than a beamwidth, a method was proposed by Capon [4]. The power spectrum is

\[ P(\phi) = \frac{1}{a^H(\phi)\hat{R}a(\phi)} \]  

(6)

Following plot shows the comparison of resolutions of CBF and MVDR

The above plot shows that using 4 sensors, even two closely spaced targets can be resolved using MUSIC beamformer. When the number of snapshots (available data is small), a modified version of MUSIC known as Root-MUSIC proves to be useful. It is based on polynomial rooting.
D. Root-MUSIC

The Root-MUSIC method converts the MUSIC spectrum into a polynomial whose solution results directly in numeric values for the estimated directions [6].

E. Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT)

This algorithm is based on translational invariance structure (e.g. ULA) of sensors [7]. ESPRIT is computationally more efficient as it doesn’t require an exhaustive search through all possible steering vectors for DOA estimation. Following figure shows the pair of subarrays used in ESPRIT.

\[ \phi_k = \sin^{-1}\left(\frac{\text{arg}(\Phi_k)}{\beta d}\right) \] (10)

where \( \phi_k \) is the estimated DOA, \( \Phi_k \) is the \( k^{th} \) eigenvalue of subspace rotational operator [7], \( \beta = 2\pi/\lambda(\text{wavelength}) \) and \( d \) is the spacing between the sensors. The following plot shows the output of ESPRIT beamformer when sources were present at \(-2^\circ, 1^\circ \) and \(4^\circ\).

IV. COMPARISON OF BEAMFORMING ALGORITHMS

Following figure shows the resolving power of different beamformers discussed.

From the above plot, it can be seen that the ESPRIT is a very high resolution algorithm.

In the following plot, the accuracy of beamformers for different SNR values are compared. The RMSE is given by

\[ \text{RMSE} = \frac{\sqrt{\text{E}(\hat{\phi}_1 - \hat{\phi}_2)^2} + \sqrt{\text{E}(\hat{\phi}_1 - \hat{\phi}_3)^2}}{2} \] (11)

It can be observed that ESPRIT is the most accurate for low SNR conditions.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Resolution</th>
<th>Complexity</th>
<th>General Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>CBF</td>
<td>Poor</td>
<td>Simple Implementation, 1-D search</td>
<td>Resolution depends on main lobe</td>
</tr>
<tr>
<td>MVDR</td>
<td>Good</td>
<td>Inverse of ( \mathbf{R} ), 1-D search</td>
<td>Poor performance in low SNR</td>
</tr>
<tr>
<td>MUSIC</td>
<td>Very Good</td>
<td>Eigenvalue Decomposition, 1-D search</td>
<td>Also estimates number of sources [5]</td>
</tr>
<tr>
<td>ESPRIT</td>
<td>Excellent</td>
<td>Eigenvalue Decomposition, Calculating ( \Psi )</td>
<td>Array needs doublets</td>
</tr>
</tbody>
</table>

V. COMPUTATIONAL COMPLEXITIES

The following tables show the number of multiplications and additions required for each of the algorithm. The following symbols are used (Table I-Table VI)

\[ N = \text{number of sensors} \]
\[ M = \text{number of signals} \]
\[ K = \text{number of samples} \]
\[ L = \text{number of angles to scan} \]
### Table I: Computation of Correlation Matrix

<table>
<thead>
<tr>
<th>Operation</th>
<th>Multiplications</th>
<th>Additions</th>
<th>Divisions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{xy} = \frac{1}{K} \sum_{n=1}^{N} x(n)x^H(n)$</td>
<td>$K \left( \frac{N^2}{2} + \frac{N}{2} \right)$</td>
<td>$(K-1)N^2$</td>
<td>$N^2$</td>
</tr>
</tbody>
</table>

### Table II: Computations of CBF

<table>
<thead>
<tr>
<th>Operation</th>
<th>Multiplications</th>
<th>Additions</th>
<th>Divisions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(\phi)<em>{x} = a^H(\phi)</em>{x,n} R_{xy}^{-1} a(\phi)_{x,n} \quad L(N^2 + N)$</td>
<td>$L(N^2 - 1)$</td>
<td>$-\quad$</td>
<td></td>
</tr>
</tbody>
</table>

### Table III: Computations of MVDR

<table>
<thead>
<tr>
<th>Operation</th>
<th>Multiplications</th>
<th>Additions</th>
<th>Divisions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{xy}^{-1} (\text{Gauss Jordan Inversion})$</td>
<td>$\frac{N^3 + N^2 + N}{3} + \frac{N^2}{2}$</td>
<td>$\frac{N^3 + N^2}{3} + \frac{N}{2}$</td>
<td>$\frac{N^2 + N}{2}$</td>
</tr>
<tr>
<td>$P(\phi)<em>{x} = \frac{1}{a^H(\phi)</em>{x,n} R_{xy}^{-1} a(\phi)_{x,n}} \quad L(N^2 + N)$</td>
<td>$L(N^2 - 1)$</td>
<td>$L\quad$</td>
<td></td>
</tr>
</tbody>
</table>

### Table IV: Computations of MUSIC Algorithm

<table>
<thead>
<tr>
<th>Operation</th>
<th>Multiplications</th>
<th>Additions</th>
<th>Divisions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{xy} = U_{N \times N} A_{N \times N} U^H_{N \times N}$</td>
<td>$\frac{16}{5} N^3$</td>
<td>$\frac{44}{5} N^3$</td>
<td>$-\quad$</td>
</tr>
<tr>
<td>$Q_{xy} = U_{N \times (N-M)} U^H_{N \times [(N-M)\times N]}$</td>
<td>$\frac{(N-M)N^2}{2} + \frac{(N-M)N}{2}$</td>
<td>$\frac{(N-M)N^2}{2} + \frac{(N-M)N}{2}$</td>
<td>$-\quad$</td>
</tr>
<tr>
<td>$P(\phi)<em>{x} = \frac{1}{a^H(\phi)</em>{x,n} Q_{xy}^{-1} a(\phi)_{x,n}} \quad L(N^2 + N)$</td>
<td>$L(N^2 - 1)$</td>
<td>$L\quad$</td>
<td></td>
</tr>
</tbody>
</table>

### Table V: Computations of ESPRIT Algorithm

<table>
<thead>
<tr>
<th>Operation</th>
<th>Multiplications</th>
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<th>Divisions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{xy} = U_{N \times N} A_{N \times N} U^H_{N \times N}$</td>
<td>$\frac{16}{5} N^3$</td>
<td>$\frac{44}{5} N^3$</td>
<td>$-\quad$</td>
</tr>
<tr>
<td>$A_{M \times M} = U_{(M \times (N-1))} U^H_{N \times [(N-1)\times M]}$</td>
<td>$\frac{(N-M)(N-1)^2}{2} + \frac{(N-M)(N-1)}{2}$</td>
<td>$\frac{(N-M)(N-1)^2}{2} + \frac{(N-M)(N-1)}{2}$</td>
<td>$-\quad$</td>
</tr>
<tr>
<td>$B_{M \times M} = A_{M \times M}^{-1} (\text{Gauss Jordan Inversion})$</td>
<td>$\frac{N^3 + N^2}{3} + \frac{N^2}{2} + \frac{N}{6}$</td>
<td>$\frac{N^3 + N^2}{3} + \frac{N^2}{2} + \frac{N}{6}$</td>
<td>$-\quad$</td>
</tr>
<tr>
<td>$C_{M \times M} = U_{(M \times (N-1))} U^H_{N \times [(N-1)\times M]}$</td>
<td>$\frac{(N-M)^2}{2} + \frac{(N-M)(N-1)}{2}$</td>
<td>$\frac{(N-M)^2}{2} + \frac{(N-M)(N-1)}{2}$</td>
<td>$-\quad$</td>
</tr>
<tr>
<td>$\Psi_{M \times M} = B_{M \times M} C_{M \times M}$</td>
<td>$(N-M)^2 - (N-M)^2$</td>
<td>$(N-M)^2 - (N-M)^2$</td>
<td>$-\quad$</td>
</tr>
</tbody>
</table>

### Table VI: Enhancement in Resolution Using Directional Sensors

When directional sensors are used instead of omnidirectional sensors, the half power beamwidth of the array’s response reduces which increases the resolution of the beamformer.

Consider an array of 4 sensors each having a linear aperture of length $D (D<d)$.
The output of CBF is shown below. The output power of each of the beamformers with directional and omnidirectional sensors is first normalized and then combined.

Similarly for MVDR and MUSIC beamformer, the increase in resolution obtained by using directional sensors is shown below.

VII. CONCLUSIONS

In this paper different algorithms for estimation of direction of arrival are discussed. The main focus is to increase the resolution so that closely spaced targets in space can be separated. The conventional beamformer has a resolution limitation due to beamwidth. It has been shown that beamwidth can be reduced by increasing the number of sensors but it also incrases the cost of beamformer. Adaptive beamforming algorithms have the advantage of much better resolution. The minimum variance distortionless response beamformer has a relatively higher resolution due to the output power minimization subject to the constraint.

Subspace methods for estimation of DOA are based on the signal and noise subspaces. MUSIC algorithm which shows high peaks for angles corresponding to DOAs, has a much higher resolution. It has also been shown that using directional sensors instead of omnidirectional sensors gives the advantages of a relatively reduced beamwidth and higher gain. The tables of computational complexities show that ESPRIT is computationally much efficient as it does not require a scan through all possible angles.

REFERENCES

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